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A numerical algorithm to find the equilibrium of a conservative system

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A numerical algorithm to find the equilibrium of a conservative system



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Why this algorithm ?

- **Aim:** determine the exact location of a dynamical equilibrium (forced oscillations only) corresponding to a resonance
- **Problem:** - accurate (realistic) simulations give some parasitical (free) librations supposed to be damped
- the initial conditions given by analytical studies are not accurate enough
- **Solution:** this algorithm use NAFF^[4] (Numerical Analysis of the Fundamental Frequencies) for the identification of the free and forced oscillations, the former being iteratively removed from the solution by carefully choosing the initial conditions.

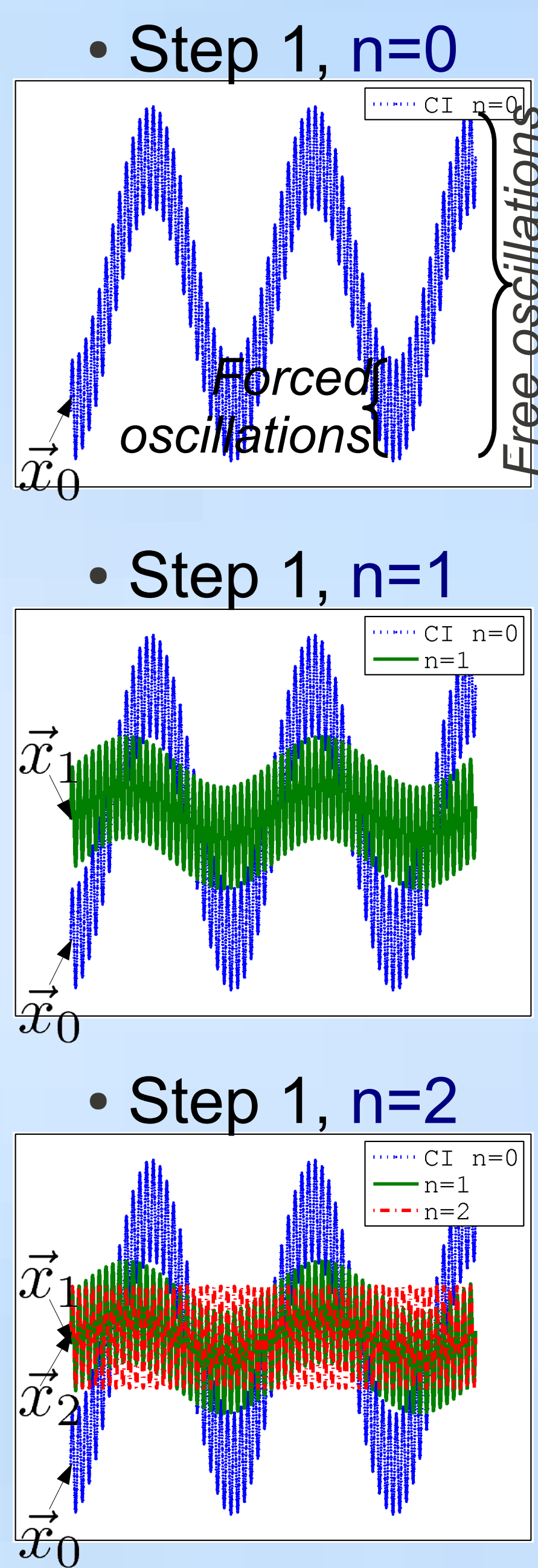
History

This algorithm has been used in some works yet:

- Rotation problem ^[3,5,7]
- Exoplanetary dynamics ^[1]
- Dynamics of a probe around Vesta ^[2]
- ...

Here^[6] we provide the convergence proof and give the quadratic convergence in the Hamiltonian case.

The Algorithm



Step 0 Initialisation

- Choose an initial condition \vec{x}_0 close to the equilibrium (e.g. using an analytical solution) and set $n=0$

Step 1 Integration

- Obtain the evolution of \vec{x} with respect to the time: $\vec{x}(t)$

Step 2 Identification

- Express the variable \vec{x} of the problem by a quasi-periodic decomposition (e.g. using NAFF^[4])
- Isolate the free (to remove) and the forced (to keep) oscillations

Step 3 New Initial Conditions

- Remove the free oscillations from the initial conditions \vec{x}_n to obtain the new initial conditions \vec{x}_{n+1}
- $$\vec{x}_{n+1} = \vec{x}_n - \sum_{i \in \text{Free terms}} \text{Ampl}_i \cos(\text{Phase}_i)$$

- Increase n: $n=n+1$
- Go to **Step 1** until $\|\vec{x}_{n+1} - \vec{x}_n\| < \varepsilon_{IC}$

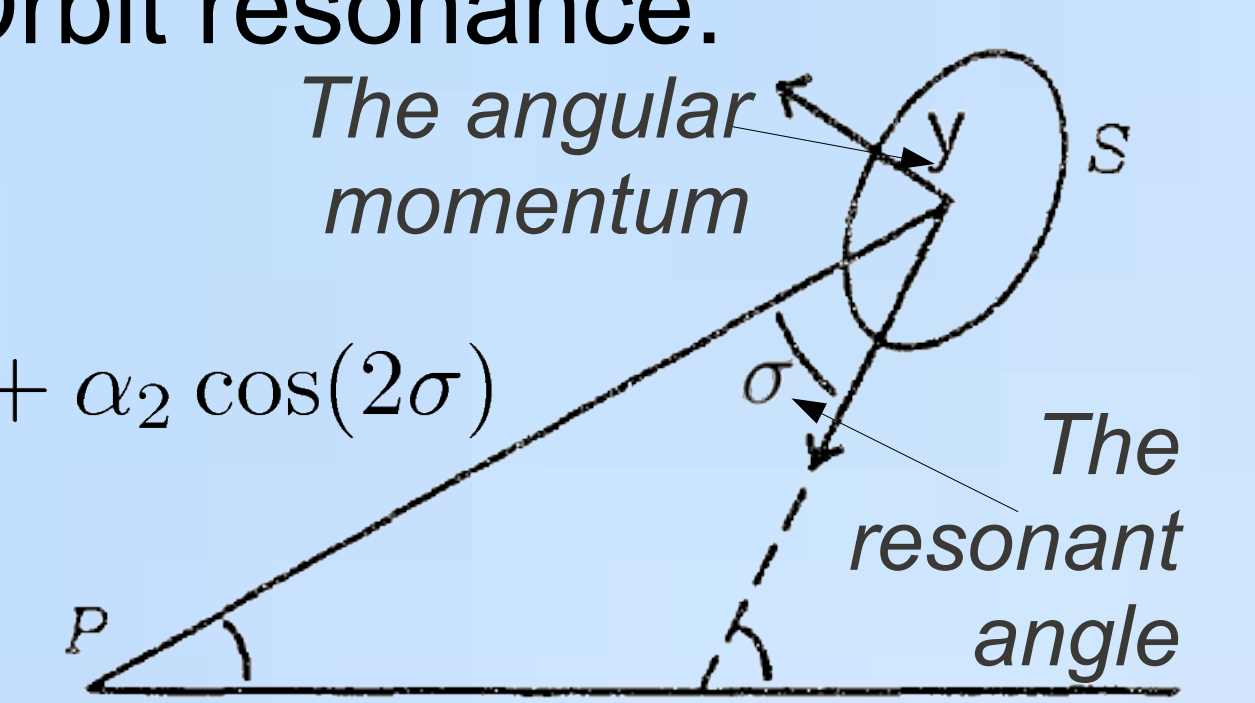
or until $\text{Ampl}_i < \varepsilon_{\text{Ampl}} \forall i$

An example

Earth-Moon Spin-Orbit resonance:

$$H(y, \sigma, L, \lambda) = \frac{y^2}{2}$$

$$- \epsilon \left[\alpha_1 \cos(2\sigma + \lambda) + \alpha_2 \cos(2\sigma) + \alpha_3 \cos(2\sigma - \lambda) + \alpha_4 \cos(2\sigma - 2\lambda) + \alpha_5 \cos(2\sigma - 3\lambda) \right] + L - y$$



- **Step 0:** the averaged Hamiltonian

$$\bar{H}(y, \sigma, L, -) = \frac{(y-1)^2}{2} - \epsilon \alpha_2 \cos(2\sigma) + L$$

gives the first initial condition: $y_0 = 0, \sigma_0 = 0$

- **Step 1** (numerical integration) + **Step 2** (NAFF) Frequency decomposition of $y(t)$

N	Ampl.	Freq.	Phase (rad)	Id.
1	1.000 000	0.000 00	2.462×10^{-14}	
2	$7.536 994 \times 10^{-4}$	1.000 00	3.141 588	λ
3	$8.010 500 \times 10^{-5}$	$2.616 86 \times 10^{-2}$	0.000 008	u
4	$4.392 058 \times 10^{-6}$	2.000 00	3.141 587	2λ
5	$3.328 440 \times 10^{-7}$	3.000 00	3.141 587	3λ

- **Step 3:** new initial conditions

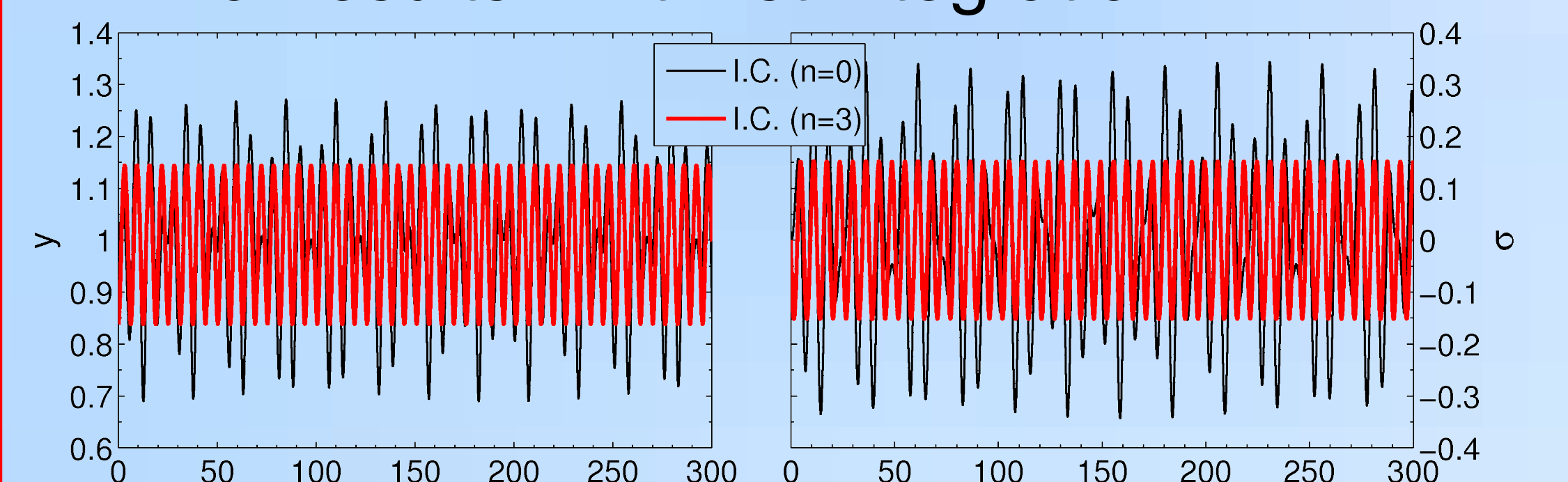
$$y_1 = y_0 - 8.01050010^{-5} \cos(0.000008) = 0.999920$$

$$\sigma_1 = \sigma_0 - 3.06110710^{-3} \cos(1.570796) = 3.10699110^{-10}$$

- Some iterations of the algorithm

n	I.C.	Ampl. - free term	ω^*
0	y 1.000 000 000	$8.010 500 \times 10^{-5}$	$2.616 864 \times 10^{-2}$
	σ 0.000 000 000	$3.061 106 \times 10^{-3}$	$2.616 864 \times 10^{-2}$
1	y 0.999 919 905	$7.770 876 \times 10^{-10}$	$2.616 870 \times 10^{-2}$
	σ $3.106 991 059 \times 10^{-10}$	$2.969 532 \times 10^{-8}$	$2.616 870 \times 10^{-2}$
2	y 0.999 919 904	$1.264 196 \times 10^{-15}$	$2.616 860 \times 10^{-2}$
	σ $3.490 283 085 \times 10^{-15}$	$4.830 976 \times 10^{-14}$	$2.616 860 \times 10^{-2}$

- Final results w.r.t first integration



Convergence proof (for Hamiltonian case)^[6]

Let $\dot{\vec{X}} = f(\vec{X}) + g(\vec{X}, t)$ with the solution

$$\phi(t; \vec{X}) = \sum_{m \in \mathbb{Z}} \phi_{0m}(\vec{X}) e^{i \nu m t} + \sum_{l \neq 0, m \in \mathbb{Z}} \phi_{lm}(\vec{X}) e^{i(\omega l + \nu m)t} := S(t; \vec{X}) + L(t; \vec{X})$$

and the fixed point \vec{X}_∞ such as $\phi(t; \vec{X}_\infty) = S(t; \vec{X}_\infty)$.

Assuming an Hamiltonian framework and an initial condition \vec{X}_0 such as $|\vec{X}_0 - \vec{X}_\infty| < 1$.

Then, the algorithm gives a sequence $(\vec{X}_n)_n$ where $\vec{X}_n \xrightarrow{n \rightarrow \infty} \vec{X}_\infty$ and the convergence rate

is quadratic $|\vec{X}_{n+1} - \vec{X}_\infty| \propto |\vec{X}_n - \vec{X}_\infty|^2$

Idea of proof (one dim. to simplify)

We have to prove that x_∞ is an attractor:

$$|f'(x_\infty)| < 1 \iff \lim_{x \rightarrow x_\infty} \frac{\partial_x S(0; x) / \partial_x L(0; f(x))}{\partial_x S(0; f(x)) / \partial_x L(0; f(x)) + 1} = 0$$

Using the d'Alembert rule we can state that (x close to x_∞)

$$S(0; x) \sim x_\infty + a|x - x_\infty| + \dots \quad \text{and} \quad L(0; x) \sim x_\infty + b\sqrt{|x - x_\infty|} + \dots$$

Then, $\partial_x S(0; x) / \partial_x L(0; f(x)) \xrightarrow{x \rightarrow x_\infty} 0$ and the convergence rate is quadratic.

References:

- [1] Couetdic J. et al (2010), Astronomy and Astrophysics, 519
- [2] Delsate N. (2011), Planetary and Space Science, 59
- [3] Dufey J. et al. (2009), Icarus, 203
- [4] Laskar J. (1993), Celestial Mechanics and Dynamical Astronomy, 56
- [5] Noyelles B. (2009), Icarus, 202
- [6] Noyelles et al., arXiv:1101.2138
- [7] Robutel P. et al. (2011), Icarus, 211